

Pseudo-stopping times and the Immersion property

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Abstract

Given two filtrations $\mathbb{F} \subset \mathbb{G}$, the immersion property holds for \mathbb{F} and \mathbb{G} if every \mathbb{F} -local martingale is a \mathbb{G} -local martingale. Our main result characterizes the immersion property for \mathbb{F} and \mathbb{G} using the class of \mathbb{F} -pseudo-stopping times. We also show that every \mathbb{G} -stopping time can be decomposed into the minimum of two barrier hitting times.

1 Introduction

The study of pseudo-stopping times started in the paper by Williams [11]. The author describes there an example of a non-stopping time τ which has the optional stopping property, namely, for every uniformly integrable martingale M , $\mathbb{E}(M_\tau) = \mathbb{E}(M_0)$. Let us recall this example here. Let B be a Brownian motion and define:

$$T_1 := \inf\{t : B_t = 1\} \quad \text{and} \quad \sigma := \sup\{t \leq T_1 : B_t = 0\}.$$

Therefore σ is the last zero of the process B before it reaches one. Let τ be the time of the maximum of B over $[0, \sigma]$, that is

$$\tau := \sup\{t < \sigma : B_t = B_t^*\} \quad \text{with} \quad B_t^* := \sup_{s \leq t} B_s.$$

Then, as shown in [11], τ has the optional stopping property. Such random times were then called pseudo-stopping times and further studied by Nikeghbali and Yor in [10].

In this paper, we study the properties of *pseudo-stopping times* in the context of the theory of enlargement of filtrations. We work on a filtered probability space $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$

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denotes a filtration satisfying the usual conditions and we set $\mathcal{F}_\infty := \bigvee_{t \geq 0} \mathcal{F}_t \subset \mathcal{A}$. A process that is not necessarily adapted to the filtration \mathbb{F} is said to be *raw*. As convention, for any martingale, we work always with its càdlàg modification, while for any random process $(X_t)_{t \geq 0}$, we set $X_{0-} = 0$ and $X_\infty = \lim_{t \rightarrow \infty} X_t$ a.s, if it exists.

Let $\mathbb{G} := (\mathcal{G}_t)_{t \geq 0}$ be another filtration such that $\mathbb{F} \subset \mathbb{G}$, that is for each $t \geq 0$, $\mathcal{F}_t \subset \mathcal{G}_t$. We say that \mathbb{F} is *immersed* in \mathbb{G} and write $\mathbb{F} \hookrightarrow \mathbb{G}$ when every \mathbb{F} -local martingale is a \mathbb{G} -local martingale. Often the immersion property is called the *hypothesis* (\mathcal{H}) in the literature. We refer the reader to Brémaud and Yor [4] for discussion and other conditions equivalent to the immersion property.

The main results of this paper are motivated by the study of the converse implications to the following known observations in the literature. Let us define a filtration $\mathbb{F}^\tau := (\mathcal{F}_t^\tau)_{t \geq 0}$ as the progressive enlargement of \mathbb{F} with τ , i.e. the smallest right-continuous filtration containing \mathbb{F} such that τ is a stopping time, that is

$$\mathcal{F}_t^\tau := \bigcap_{s > t} (\mathcal{F}_s \vee \sigma(\tau \wedge s)).$$

In the reduced form approach to credit risk modelling (see Bielecki et al. [3]), given a filtration \mathbb{F} , a popular way to model the default time τ is to use a barrier hitting time of an \mathbb{F} -adapted increasing process and an independent barrier. It is known that a random time constructed in this fashion has the property that the filtration \mathbb{F} is immersed in \mathbb{F}^τ . It is also known that the property that $\mathbb{F} \hookrightarrow \mathbb{F}^\tau$ implies that every \mathbb{F}^τ -stopping time is an \mathbb{F} -pseudo stopping time.

The authors of [10] have remarked that, given two filtrations \mathbb{F} and \mathbb{G} such that \mathbb{F} is immersed in \mathbb{G} , every \mathbb{G} -stopping time is an \mathbb{F} -pseudo-stopping time. The main result of this work is Theorem 2, where we first note that the converse is true: if every \mathbb{G} -stopping time is \mathbb{F} -pseudo-stopping time then \mathbb{F} is immersed in \mathbb{G} . This provides an alternative characterization of the immersion property based on pseudo-stopping times. Further we prove third equivalent statement involving \mathbb{F} -(dual) optional projections of \mathbb{G} -adapted processes with finite variation. As an application of Theorem 2, we provide in Proposition 3 an alternative proof to a result regarding the immersion property and the progressive enlargement with honest times. The advantage of our method is that we do not use specific structures of the progressive enlargement and the characterization of predictable sets as done in Jeulin [9].

Assuming that \mathbb{F} is immersed in \mathbb{G} , we show in Theorem 4 that every \mathbb{G} -stopping time can be written as the minimum of two \mathbb{G} -stopping times. One of which is a barrier hitting time of an \mathbb{F} -adapted increasing process, where the barrier is '*almost*' independent, and the other is an \mathbb{F} -pseudo-stopping time whose graph is contained in the union of the graphs of a family of \mathbb{F} -stopping times.

2 Main results

2.1 Pseudo-stopping times and the immersion property

In this section, we give a new characterization of the immersion property in terms of pseudo-stopping times and dual optional projections. As an application, we give a new proof to a result from the theory of enlargement of filtrations involving honest times.

The main object of interest is the class of pseudo-stopping times. We first recall the definition of pseudo-stopping times from [10], with a slight modification, that is the random time is allowed to take the value infinity.

Definition 1. A random time τ is an *\mathbb{F} -pseudo-stopping time* if for every uniformly integrable \mathbb{F} -martingale M , we have $\mathbb{E}(M_\tau) = \mathbb{E}(M_0)$.

The main result is the following.

Theorem 2. *Given two filtrations \mathbb{F} and \mathbb{G} such that $\mathbb{F} \subset \mathbb{G}$, the following are equivalent*

- (i) *the filtration \mathbb{F} is immersed in \mathbb{G} ;*
- (ii) *every \mathbb{G} -stopping time is an \mathbb{F} -pseudo-stopping time;*
- (iii) *the \mathbb{F} -dual optional projection of any \mathbb{G} -optional process of integrable variation is equal to its \mathbb{F} -optional projection.*

An important class of random times is a class of honest times. A random time τ is an \mathbb{F} -honest time if for every $t > 0$ there exists an \mathcal{F}_t -measurable random variable τ_t such that $\tau = \tau_t$ on $\{\tau < t\}$.

In Proposition 3 we relate pseudo-stopping times with honest times and, as an application of our main result in Theorem 2, we recover a new proof of a result regarding honest times and the immersion property found in Jeulin [9]. Therein the result is obtained by computing explicitly the \mathbb{G} -semimartingale decompositions of \mathbb{F} -martingales. The following Proposition was presented in Proposition 6 in [10] under the simplifying assumption that all \mathbb{F} -martingales are continuous and the proof therein uses distributional arguments. Here, we show that similar result can be obtained in full generality by using sample path properties and Theorem 6 (v).

Proposition 3. *Let τ be a random time. Then, the following conditions are equivalent*

- (i) *τ is equal to an \mathbb{F} -stopping time on $\{\tau < \infty\}$;*
 - (ii) *τ is an \mathbb{F} -pseudo-stopping time and an \mathbb{F} -honest time.*
- In particular if τ is an \mathbb{F} -honest time which is not equal to an \mathbb{F} -stopping time on $\{\tau < \infty\}$ and a \mathbb{G} -stopping time for some filtration $\mathbb{G} \supset \mathbb{F}$ then \mathbb{F} is not immersed in \mathbb{G} .*

2.2 Characterization of \mathbb{G} -stopping times

In this section, given that $\mathbb{F} \hookrightarrow \mathbb{G}$, we characterize all \mathbb{G} -stopping times in terms of barrier hitting times of \mathbb{F} -adapted increasing processes, where the \mathcal{F}_∞ -conditional distribution of the barrier can be computed. The main result of this section is given in Theorem 4.

The main tools used in this study are the (dual) optional projections onto the filtration \mathbb{F} . We record here some known results from the general theory of stochastic processes. For more details of the theory the reader is referred to He et al. [7] or Jacod and Shiryaev [8] and for specific results from the theory of enlargement of filtrations to Jeulin [9].

For any locally integrable variation process V , we denote the \mathbb{F} -optional projection of V by oV and the \mathbb{F} -dual optional projection of V by V^o . It is known that the process $N^V := {}^oV - V^o$ is a uniformly integrable \mathbb{F} -martingale with $N_0^V = 0$ and ${}^o(\Delta V) = \Delta V^o$.

We specialize the above notions to the study of random times. For an arbitrary random time τ , we set $A := \mathbb{1}_{[\tau, \infty]}$ and define

- the supermartingale Z associated with τ , $Z := {}^o(\mathbb{1}_{[0, \tau]}) = 1 - {}^oA$,
- the supermartingale \tilde{Z} associated with τ , $\tilde{Z} := {}^o(\mathbb{1}_{[0, \tau]}) = 1 - {}^o(A_-)$,
- the martingale $m := 1 - ({}^oA - A^o)$.

These processes are linked through the following relationships:

$$Z = m - A^o \quad \text{and} \quad \tilde{Z} = m - A_-^o.$$

We present now our final result, which shows that under the assumption that $\mathbb{F} \hookrightarrow \mathbb{G}$, every \mathbb{G} -stopping time can be written as the minimum of two barrier hitting times for which the \mathcal{F}_∞ -conditional distribution of the barriers can be computed.

Theorem 4. *Assume that $\mathbb{F} \hookrightarrow \mathbb{G}$ and let τ be a \mathbb{G} -stopping time. Then τ can be written as $\tau_c \wedge \tau_d$, where:*

- (i) *The random time τ_c is a \mathbb{G} -stopping time which avoids all finite \mathbb{F} -stopping times. Denote by $A^{c,o}$ the \mathbb{F} -dual optional projection of the process $\mathbb{1}_{[\tau_c, \infty]}$. Then the \mathcal{F}_∞ -conditional distribution of $A_{\tau_c}^{c,o}$ is uniform on the interval $[0, A_\infty^{c,o}]$, with an atom of size $1 - A_\infty^{c,o}$ at $A_\infty^{c,o}$, that is*

$$\mathbb{P}(A_{\tau_c}^{c,o} \leq u \mid \mathcal{F}_\infty) = u \mathbb{1}_{\{u < A_\infty^{c,o}\}} + \mathbb{1}_{\{u \geq A_\infty^{c,o}\}}.$$

(ii) The random time τ_d is a \mathbb{G} -stopping time whose graph is contained in the disjoint union of the graphs of the jump times of the process A° given by $(\sigma_k)_{k \in \mathbb{N}}$. Denote by $A^{d,\circ}$ the \mathbb{F} -dual optional projection of the process $\mathbb{1}_{[\tau_d, \infty[}$. Then

$$\mathbb{P}(A_{\nu_d}^{d,\circ} = u \mid \mathcal{F}_\infty) = \sum_k \mathbb{1}_{\{A_{\sigma_k}^{d,\circ} = u\}} \Delta A_{\sigma_k}^\circ.$$

Remark 5. As a special case of the above results, if τ is a finite \mathbb{G} -stopping time that avoids finite \mathbb{F} -stopping times, then A_τ° is independent of \mathcal{F}_∞ and uniformly distributed on the interval $[0, 1]$. In this case, the \mathbb{G} -stopping time τ is a barrier hitting time of an \mathbb{F} -adapted increasing process, with the barrier being independent from \mathcal{F}_∞ . This is a class of random times widely used in credit risk modelling to model default times.

3 Proofs & complementary results on pseudo-stopping times

3.1 Complementary results on pseudo-stopping times

For our purposes, we present in Theorem 6 an extension of Theorem 1 from Nikeghbali and Yor [10]. We extend their result in two directions. Firstly, we allow for non-finite pseudo-stopping times. Secondly, Theorem 1 from [10] states that if either all \mathbb{F} -martingales are continuous or the random time τ avoids all finite \mathbb{F} -stopping times, i.e. $\Delta A^\circ = 0$, then the random time τ is an \mathbb{F} -pseudo-stopping time if and only if the process Z is a decreasing \mathbb{F} -predictable process. We will remove these additional assumptions and present another equivalent characterization based on the process \tilde{Z} instead of Z in condition (v) of Theorem 6.

Theorem 6. *The following are equivalent:*

- (i) τ is an \mathbb{F} -pseudo-stopping time;
- (ii) $A_\infty^\circ = \mathbb{P}(\tau < \infty \mid \mathcal{F}_\infty)$;
- (iii) $m = 1$ or equivalently ${}^\circ A = A^\circ$;
- (iv) for every \mathbb{F} -local martingale M , the process M^τ is an \mathbb{F}^τ -local martingale;
- (v) the process \tilde{Z} is a càglàd decreasing process.

Another motivation of this work is to better understand the property that ${}^\circ A = A^\circ$. In essence, this property says that the optional projection is equal to the dual optional projection, which is not true in general.

Before proceeding to the proof of Theorem 6, we give an auxiliary lemma which characterizes the main property of our interest, that is, given a process of finite variation, when is its optional projection equal to the dual optional projection.

Lemma 7. *Given a raw locally integrable increasing process V , the following are equivalent:*

- (i) ${}^\circ(V_-)$ is a càglàd increasing process or equivalently ${}^\circ(V_-) = V_-^\circ$;
- (ii) ${}^\circ(V_-) = {}^\circ V_-$;
- (iii) ${}^\circ V = V^\circ$.

Proof. For any raw locally integrable increasing process V , from classic theory we know that the process $N^V := {}^\circ V - V^\circ$ is a uniformly integrable martingale with $N_0^V = 0$ and ${}^\circ(\Delta V) = \Delta V^\circ$. As a consequence we have

$$N^V = {}^\circ(V_-) - V_-^\circ \quad \text{and} \quad N_-^V = {}^\circ V_- - V_-^\circ. \quad (1)$$

If ${}^\circ(V_-)$ is a càglàd increasing process, then from (1), we see that N^V is a predictable martingale of finite variation, therefore is constant and equal to zero, since predictable martingales are continuous which shows (i) \implies (iii). For the converse, it is enough to use the definition of N^V . Since N^V is càdlàg, we know that $N^V = 0$ if and only if $N_-^V = 0$. This fact combined with (1) gives the equivalence between (i) and (ii). \square

Proof of Theorem 6. To see that (i) and (ii) are equivalent, suppose τ is an \mathbb{F} -pseudo-stopping time. Then, by properties of optional and dual optional projection, for any uniformly integrable \mathbb{F} -martingale M we have

$$\mathbb{E}(M_\tau \mathbf{1}_{\{\tau < \infty\}}) = \mathbb{E}\left(\int_{[0, \infty)} M_s dA_s^o\right) = \mathbb{E}(M_\infty A_\infty^o).$$

Therefore, the equality, $\mathbb{E}(M_\tau) = \mathbb{E}(M_\infty)$ holds true for every uniformly integrable \mathbb{F} -martingale M if and only if $A_\infty^o = \mathbb{P}(\tau < \infty | \mathcal{F}_\infty)$, since $\mathbb{E}(M_\tau) = \mathbb{E}(M_\infty (A_\infty^o + \mathbb{P}(\tau = \infty | \mathcal{F}_\infty)))$.

On the other hand, ${}^oA_\infty = \lim_{s \rightarrow \infty} \mathbb{P}(\tau \leq s | \mathcal{F}_s) = \mathbb{P}(\tau < \infty | \mathcal{F}_\infty)$ a.s., and from the definition of m , we note that (ii) holds if and only if (iii) holds, that is $m = 1$ or equivalently ${}^oA = A^o$. The equivalence of (iii) and (v) follows directly from Lemma 7.

To see that (i) \implies (iv), let M be a uniformly integrable \mathbb{F} -martingale. For any \mathbb{F}^τ -stopping time ν , from Dellacherie et al. [5] (page 186), we know there exists an \mathbb{F} -stopping time σ such that $\rho \wedge \nu = \rho \wedge \sigma$. Therefore, from the definition of pseudo-stopping time,

$$\mathbb{E}(M_{\rho \wedge \nu}) = \mathbb{E}(M_{\rho \wedge \sigma}) = \mathbb{E}(M_0),$$

which shows that M is a uniformly integrable \mathbb{F}^τ -martingale by Theorem 1.42 [8]. The implication (iv) \implies (i) is straightforward. \square

In the following example, taken from Proposition 5.3 in [2], we illustrate the importance of the càglàd property in condition (v) of Theorem 6. From this example we also see that a decreasing supermartingale Z is not sufficient to ensure that the time is a pseudo-stopping time. We would also like to point out that condition (v) in Theorem 6 is crucial when working with non-continuous filtrations and it is used in the proof of Proposition 3.

Example 8. Let N be a Poisson process with intensity λ and jump times $(T_n)_n$. Consider the random time $\tau = \frac{1}{2}(T_1 + T_2)$. Then

$$\tilde{Z}_t = Z_t = \mathbf{1}_{\{T_1 > t\}} + \mathbf{1}_{\{T_1 \leq t\}} \mathbf{1}_{\{T_2 > t\}} e^{-\lambda(t - T_1)}$$

and

$$m_t = - \int_0^t e^{-\lambda(s - T_1)} dM_s \quad \text{with} \quad M_s = \mathbf{1}_{\{T_2 \leq s\}} - (\lambda(s \wedge T_2) - \lambda(s \wedge T_1)).$$

Therefore, m is not a constant martingale and τ is not a pseudo-stopping time but $\tilde{Z} = Z$ is a decreasing and càdlàg process.

3.2 Proofs of Theorem 2 and Proposition 3

Proof of Theorem 2. To show (i) \implies (ii), let M be any uniformly integrable \mathbb{F} -martingale and ν a \mathbb{G} -stopping time. Then, from immersion property, M is a uniformly integrable \mathbb{G} -martingale and $\mathbb{E}(M_\nu) = \mathbb{E}(M_0)$, which implies ν is an \mathbb{F} -pseudo-stopping time.

To show (ii) \implies (i), suppose that M is a uniformly integrable \mathbb{F} -martingale and ν is any \mathbb{G} -stopping time. Since every \mathbb{G} -stopping time is an \mathbb{F} -pseudo-stopping time, we have $\mathbb{E}(M_\nu) = \mathbb{E}(M_0)$ for every \mathbb{G} -stopping time ν , which by Theorem 1.42 in [8], implies that M is a uniformly integrable \mathbb{G} -martingale.

The implication (iii) \implies (ii) follows directly from Theorem 6 (iii), therefore we show only the implication (i) \implies (iii). Under the immersion property, the \mathbb{F} -optional projection of any bounded \mathbb{G} -optional process is equal to its optional projection on to the constant filtration \mathcal{F}_∞ (see Bremaud and Yor [4]). More explicitly, for any given locally integrable increasing \mathbb{G} -adapted process V , we have ${}^o(V_-)_\sigma = \mathbb{E}(V_{\sigma-} | \mathcal{F}_\infty)$ for any \mathbb{F} -stopping time σ . From this we see that the process ${}^o(V_-)$ is increasing càglàd and (iii) follows from Lemma 7. \square

Proof of Proposition 3. The implication (i) \implies (ii) is obvious so we show only (ii) \implies (i). Given that τ is a honest time, by Proposition 5.2. in [9], we have that $\tau = \sup\{t : \tilde{Z}_t = 1\}$ on $\{\tau < \infty\}$. On the other hand, by Theorem 6 (v), the pseudo-stopping time property of τ implies that $\tilde{Z} = 1 - A^\circ_-$. Therefore, on $\{\tau < \infty\}$,

$$\tau = \sup\{t : \tilde{Z}_t = 1\} = \sup\{t : A^\circ_{t-} = 0\} = \inf\{t : A^\circ_t > 0\},$$

so, τ is equal to an \mathbb{F} -stopping time on $\{\tau < \infty\}$.

Therefore if τ is an \mathbb{F} -honest time which is not equal to an \mathbb{F} -stopping time on $\{\tau < \infty\}$ and a \mathbb{G} -stopping time for some filtration $\mathbb{G} \supset \mathbb{F}$ then, by Theorem 2, \mathbb{F} is not immersed in \mathbb{G} . \square

3.3 Proof of Theorem 4

Before proceeding to the proof of Theorem 4, we show that in fact any random time τ can be written as a barrier hitting time of an \mathbb{F} -adapted increasing process given the appropriate barrier. We refer the reader to Remark 3.2 in Gapeev [6] where the author considers the situation where the process A° is strictly increasing. We will demonstrate this result with no assumptions on A° .

Lemma 9. *A random time τ can be written as the barrier hitting time of the process A° with the barrier A°_τ , that is $\tau = \inf\{t > 0 : A^\circ_t \geq A^\circ_\tau\}$.*

Proof. We first define another random time τ^* by setting

$$\tau^* := \inf\{t > 0 : A^\circ_t \geq A^\circ_\tau\}.$$

To see that $\tau^* = \tau$ (it is obvious that $\tau^* \leq \tau$), we use Lemma 4.2 of [9] which states that the left-support of the measure dA , i.e.,

$$\{(\omega, t) : \forall \varepsilon > 0 \quad A_t(\omega) > A_{t-\varepsilon}(\omega)\} = \llbracket \tau \rrbracket$$

belongs to the left-support of dA° , i.e., to the set $\{(\omega, t) : \forall \varepsilon > 0 \quad A^\circ_t(\omega) > A^\circ_{t-\varepsilon}(\omega)\}$. \square

Proof of Theorem 4. For any \mathbb{G} -stopping time τ and the set $D := \{\Delta A^\circ_\tau > 0\} \in \mathcal{G}_\tau$, we see that τ can be written as $\tau_c \wedge \tau_d$, where $\tau_c := \tau \mathbf{1}_{D^c} + \infty \mathbf{1}_D$ and $\tau_d := \tau \mathbf{1}_D + \infty \mathbf{1}_{D^c}$. The random times τ_c and τ_d are therefore \mathbb{G} -stopping times, where τ_c avoids finite \mathbb{F} -stopping times and the graph of τ_d is contained in the graphs of the jump times of A° . For more details on this decomposition of a random time see [1].

Given τ is an \mathbb{G} -stopping time that avoids all finite \mathbb{F} -stopping times. The \mathcal{F}_∞ -conditional distribution of A°_τ is given by

$$\mathbb{E}_\mathbb{P}(\mathbf{1}_{\{A^\circ_\tau \leq u\}} \mid \mathcal{F}_\infty) = \mathbb{E}_\mathbb{P}(\mathbf{1}_{\{A^\circ_\tau \leq u\}} \mid \mathcal{F}_\infty) \mathbf{1}_{\{u < A^\circ_\infty\}} + \mathbf{1}_{\{u \geq A^\circ_\infty\}}.$$

Let us set C to be the right inverse of A° , then the first term in the right hand side above is

$$\begin{aligned} \mathbb{E}_\mathbb{P}(\mathbf{1}_{\{A^\circ_\tau \leq u\}} \mathbf{1}_{\{C_u < \infty\}} \mid \mathcal{F}_\infty) &= \mathbb{E}_\mathbb{P}(\mathbf{1}_{\{\tau \leq C_u\}} \mathbf{1}_{\{C_u < \infty\}} \mid \mathcal{F}_{C_u}) \\ &= {}^o A_{C_u} \mathbf{1}_{\{C_u < \infty\}} \\ &= A^\circ_{C_u} \mathbf{1}_{\{C_u < \infty\}} \\ &= u \mathbf{1}_{\{u < A^\circ_\infty\}} \end{aligned}$$

where we apply Theorem 2 in the third equality, while last equality follows from the fact that $A^\circ_{C_u} = u$, since A° is continuous except perhaps at infinity. This implies that the \mathcal{F}_∞ -conditional distribution of A°_τ is uniform on $[0, A^\circ_\infty)$.

On the other hand, given τ is an \mathbb{G} -stopping time whose graph is contained in the graphs of the jump times of A^o given by $(\sigma_k)_{k \in \mathbb{N}}$. Then

$$\begin{aligned} \mathbb{P}(A_\tau^o = u \mid \mathcal{F}_\infty) &= \sum_k \mathbb{P}(\{\tau = \sigma_k\} \cap \{A_{\sigma_k}^o = u\} \mid \mathcal{F}_\infty) \\ &= \sum_k \mathbb{1}_{\{A_{\sigma_k}^o = u\}} \mathbb{P}(\tau = \sigma_k \mid \mathcal{F}_\infty) \\ &= \sum_k \mathbb{1}_{\{A_{\sigma_k}^o = u\}} \Delta A_{\sigma_k}^o \end{aligned}$$

where the last equality follows from the fact that $\mathbb{F} \hookrightarrow \mathbb{G}$. \square

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References

- [1] Aksamit, A., Choulli, T. and Jeanblanc, M.: *Decomposition of the random time*. Working paper. 2015.
- [2] Aksamit, A., Choulli, T., Deng, J. and Jeanblanc, M.: *Arbitrages in a Progressive Enlargement Setting*. In Arbitrage, credit and informational risks, ed C. Hillairet, M. Jeanblanc, Y. Jiao. p. 53-86. 2014.
- [3] Bielecki, T., Jeanblanc, M. and Rutkowski, M.: *Credit Risk Modeling*. Osaka University CSFI Lecture Notes Series 2, Osaka University Press. 2009.
- [4] Brémaud, P. and Yor, M.: *Changes of filtrations and of probability measures*. Probability Theory and Related Fields 45, p. 269-295. 1978.
- [5] Dellacherie, C., Maisonneuve, B. and Meyer, P. A.: *Probabilités et Potentiel*, Vol. 5, Herman. 1992.
- [6] Gapeev, P V.: *Some extensions of Norros' lemma in models with several defaults*. Inspired by Finance, The Musiela Festschrift. Kabanov Yu. M., Rutkowski M., Zariphopoulou Th. eds. Springer p. 273-281. 2014.
- [7] He, S.W., Wang, J.G. and Yan. J.A.: *Semimartingale theory and stochastic calculus*, Science Press. Boca Raton, FL: CRC Press Inc.. 1992.
- [8] Jacod, J. and Shiryaev, A.N.: *Limit Theorems for Stochastic Processes*. Springer, Berlin Heidelberg New York. 2003.
- [9] Jeulin, T.: *Semi-martingales et grossissement d'une filtration. Lecture Notes in Mathematics 833*. Springer, Berlin Heidelberg New York. 1980.
- [10] Nikeghbali, A. and Yor, M.: *A definition and some characteristic properties of pseudo-stopping times*. Ann. Prob. 33, p. 1804-1824. 2005.
- [11] Williams, D.: *A 'non-stopping' time with the optional stopping property*. Bulletin of the London Mathematical Society, 34, p. 610-612. 2002.